Asymptotic self-similarity breaking at late times in cosmology.

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Abstract

      We study the late time evolution of a class of exact anisotropic cosmological solutions of Einstein's equations, namely spatially homogeneous cosmologies of Bianchi type VII_0 with a perfect fluid source. We show that, in contrast to models of Bianchi type VII_h which are asymptotically self-similar at late times, Bianchi VII_0 models undergo a complicated type of self-similarity breaking. This symmetry breaking affects the late time isotropization that occurs in these models in a significant way: if the equation of state parameter γ satisfies $\gamma \leq \frac{4}{3}$ the models isotropize as regards the shear but not as regards the Weyl curvature. Indeed these models exhibit a new dynamical feature that we refer to as *Weyl curvature dominance*: the Weyl curvature dominates the dynamics at late times. By viewing the evolution from a dynamical systems perspective we show that, despite the special nature of the class of models under consideration, this behaviour has implications for more general models.

1   Introduction

In cosmology there are two asymptotic regimes, namely

  i) the approach to the initial singularity, characterized by $l \to 0$, and

  ii) the late time evolution, characterized by $l \to \infty$,

where l is the overall length scale. In other words, the asymptotic regimes correspond to the extreme values of the length scale variable. Alternatively, one can characterize the asymptotic regimes in terms of the Hubble variable H, which is related to l by

$$H = \frac{\dot{l}}{l}, \tag{1.1}$$

where the overdot denotes differentiation along the fundamental congruence. For an ever-expanding universe, $H > 0$ throughout the evolution, and the singular regime is characterized by $H \to +\infty$ and the late-time regime by $H \to 0$.

    The two asymptotic regimes are of course totally different from a physical point of view, since at the singularity physical quantities (for example, the matter density, the shear of the Hubble flow, the Weyl curvature) typically diverge, while at late times physical quantities typically tend to zero. Despite this physical distinction, the asymptotic regimes of the simplest ever-expanding models, the isotropic Friedmann-Lemaître models and the anisotropic but spatially homogeneous models of Bianchi type I, have an important property in common: *they are asymptotically self-similar* [1], that is, they are approximated by self-similar models in the asymptotic regime, as was first pointed out by Eardley (1974, page 304).

    The above behaviour becomes less surprising in view of the following heuristic consideration: in both asymptotic regimes, one has a self-gravitating system evolving in size through many orders of magnitude (evolution into the past for the singular regimes and into the

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[1] The flat FL model is in fact (exactly) self-similar (Eardley 1974, page 304).

future for the late time regime), making it plausible that the system might become scale-invariant i.e. self-similar, asymptotically. It is thus tempting to speculate that a "Principle of Asymptotic Self-Similarity" holds quite generally for Bianchi universes, namely that any Bianchi universe is approximated by a self-similar Bianchi universe in the asymptotic regimes. However, the well-known Mixmaster universes (Bianchi types VIII & IX) provide a decisive counter-example to the validity of such a principle for the singular regime, since they oscillate indefinitely as the initial singularity is approached into the past, and thus do not have a well-defined asymptotic state. Nevertheless the Mixmaster universes reinforce the idea that self-similar models are important as regards the overall evolution of Bianchi universes, since they evolve through an infinite sequence of Kasner (i.e. self-similar) states as they approach the initial singularity into the past. Indeed, research to date strongly suggests that a non-tilted perfect fluid Bianchi universe either approaches a unique self-similar asymptotic state, or it passes through an infinite sequence of self-similar states as one follows the evolution into the past towards the initial singularity.

The question remains as to the validity of the Principle of Asymptotic Self-Similarity in the late-time regime. *Our main goal in this paper is to show that the non-tilted Bianchi VII$_0$ universes provide a counter-example to asymptotic self-similarity in the late time regime.*

One can gain a deeper understanding of asymptotic self-similarity in Bianchi universes and of the reasons for its violation by viewing the Einstein field equations from a dynamical systems perspective. In studying physical phenomena, it is usually desirable to use dimensionless variables. In cosmology the density parameter, a dimensionless quantity defined by [2]

$$\Omega_0 = \left(\frac{8\pi G\mu}{3H^2}\right)_0, \tag{1.2}$$

where μ is the matter density, G is the gravitational constant and a subscript zero means evaluation at the present time, is one of the fundamental observational parameters. In addition the dimensionless quantity $(\sigma/H)_0$, where σ is the shear scalar of the Hubble flow, gives a measure of the anisotropy in the Hubble flow. This quantity is constrained by the observations of the anisotropy in the temperature of the cosmic microwave background radiation (CMBR).

In view of these considerations, it is natural to formulate the Einstein field equations using dimensionless variables that are defined by normalizing in an appropriate way with the Hubble scalar H, as in Wainwright and Hsu 1989. The appropriate time variable is a dimensionless quantity τ that is related to the length scale l by

$$l = l_0 e^\tau, \tag{1.3}$$

where l_0 is a constant. We shall refer to τ as *the dimensionless time variable.* In terms of τ, the singular regime is defined by $\tau \to -\infty$ and the late time regime for an ever-expanding model by $\tau \to +\infty$. The dimensionless time τ is related to clock time t by

$$\frac{dt}{d\tau} = \frac{1}{H}, \tag{1.4}$$

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[2]From now on we will use geometrized units, i. e. $8\pi G = 1$ and $c = 1$

as follows from equations (1.1) and (1.3). We refer to Wainwright and Ellis 1997 (Chapters 5 and 6) for full details [3].

For Bianchi universes the Einstein field equations with a perfect fluid source and an equation of state $p = (\gamma - 1)\mu$ reduce to an autonomous system of ordinary differential equations, thereby defining a dynamical system. It turns out that if one uses the above dimensionless variables, the equilibrium points (i.e. fixed points) of this dynamical system correspond to self-similar Bianchi universes, provided that $\gamma > 0$. Within this framework, the evolution of a Bianchi universe is described by an orbit of the dynamical system, and asymptotic self-similarity into the past/future means that the orbit approaches an equilibrium point into the past/future. Conversely, *asymptotic self-similarity is violated whenever the orbit of a cosmological model does not approach a single equilibrium point into the past/future.*

In the Mixmaster model the asymptotic self-similarity breaking is due to the fact that into the past, the orbits approach a bounded two-dimensional attractor containing the set of Kasner equilibrium points (the so-called Mixmaster attractor, see WE, page 146), instead of a single equilibrium point. On the other hand, we will show that the breaking of asymptotic self-similarity in the late time regime in Bianchi VII$_0$ models is due to the fact that state space is unbounded, and the orbits "escape to infinity" at late times. The difference between the two types of symmetry breaking is reflected in the behaviour of dimensionless scalars formed by normalizing with the Hubble scalar H. In a self-similar model such scalars are constant. It follows that in an asymptotically self-similar regime the limits of such scalars exist. In a Mixmaster universe the limits of some scalars of this type do not exist in the singular asymptotic regime (see WE, figures 11.1-11.3), but all remain bounded. On the other hand, in a Bianchi VII$_0$ late time regime we will show that, depending on the equation of state of the perfect fluid, a dimensionless scalar constructed from the Weyl curvature tensor oscillates and becomes unbounded.

In section 2 we introduce the three primary dimensionless scalars and summarize their behaviour in the late time asymptotic regime. This section contains no technical details, and is intended to give an overview of the main results. In Section 3 we introduce the basic dimensionless variables, which are based on the orthonormal frame formalism, referring to WE (chapters 5 and 6) for full details. A change of variable then leads to a new form of the evolution equations that is adapted to the oscillatory nature of the Bianchi VII$_0$ models. The detailed asymptotic form of the solutions of these evolution equations in the late time regime is given (see theorems 3.1 - 3.3). Section 4 contains a discussion of the cosmological implications of our results.

There are three Appendices. Appendix A contains a proof of the fact that Bianchi VII$_0$ universes are not asymptotically self-similar at late times, while Appendix B contains the lengthy proof of Theorem 3.1, which gives the asymptotic form of the solutions when the equation of state parameter satisfies $\frac{2}{3} < \gamma < \frac{4}{3}$. In Appendix C we use the asymptotic expansions in Theorems 3.1 - 3.3 to determine the asymptotic form of the line-element, so as to be able to compare our results with those of other investigations. We note, however, that knowledge of the orthonormal frame components is sufficient for analyzing any aspect of the model, for example the degree of isotropization and the anisotropy of the cosmic microwave background radiation. The asymptotic form of the line-element is thus not central to our

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[3]This work will henceforth be referred to as WE.

conclusions, and hence is placed in an appendix.

2   Main results

Our main results concern the dynamics in the late time regime of non-tilted perfect fluid Bianchi universes of group type VII_0. The Bianchi VII_0 universes that are locally rotationally symmetric [4] either admit a group of Bianchi type I or are flat FL universes, and hence are asymptotically self-similar. We thus exclude this special class in the sequel. The first result concerns the breaking of asymptotic self-similarity.

Theorem 2.1:
If $\frac{2}{3} < \gamma < 2$, then any non-tilted perfect fluid Bianchi universe of group type VII_0 that is not LRS is not asymptotically self-similar as $\tau \to +\infty$.

The proof of this Theorem, which requires some technical results from the theory of dynamical systems, is given in Appendix A.

As mentioned in the Introduction, the Hubble scalar H is used to define dimensionless variables in cosmology. We recall that H is defined by $H = \frac{1}{3}u^a_{;a}$, where $\mathbf{u}$ is the 4-velocity of the Hubble flow (the cosmological fluid), and determines the lenth scale l according to (1.1). If a Bianchi universe is asymptotically self-similar at late times, then all dimensionless scalars formed by normalizing with the Hubble scalar H have limits as $\tau \to +\infty$. As a result of Theorem 2.1, it is possible that the limits of some dimensionless scalars will not exist as $\tau \to +\infty$. The next results give the limiting behaviour of three physically important dimensionless scalars.

The first scalar is the *density parameter* Ω, defined by [5]

$$\Omega = \frac{\mu}{3H^2}, \tag{2.1}$$

where μ is the matter density of the perfect fluid. The density parameter gives a measure of the dynamical significance of the matter content of the model. If $\Omega \ll 1$ then the dynamics will be close to a vacuum model, characterized by $\Omega = 0$.

The second scalar is the *shear parameter* Σ defined by

$$\Sigma^2 = \frac{\sigma^2}{3H^2}, \tag{2.2}$$

where $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$, and σ_{ab} is the rate-of-shear tensor of the Hubble flow. The shear parameter gives a dimensionless measure of the anisotropy in the Hubble flow by comparing the shear scalar σ to the overall rate of expansion as described by H. The anisotropy in the temperature of the CMBR enables one to estimate the value of Σ at the present epoch (see for example Maartens et. al. 1996).

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[4]See for example WE, pages 22-3; we use the standard abbreviation LRS.

[5]It is customary to use a subscript zero on scalar such as Ω and H, as in equation (1.2), to indicate their present day value.

The third scalar gives a dimensionless measure of the Weyl curvature tensor, and is defined by

$$\mathcal{W} = \frac{W}{H^2},\tag{2.3}$$

where

$$W^2 = \tfrac{1}{6}\left(E_{ab}E^{ab} + H_{ab}H^{ab}\right).\tag{2.4}$$

Here E_{ab} and H_{ab} are the electric and magnetic parts of the Weyl tensor, defined by

$$E_{ab} = C_{arbs}u^r u^s, \quad H_{ab} = {}^* C_{arbs}u^r u^s,$$

where $\mathbf{u}$ is the 4-velocity of the Hubble flow and * denotes the dual operation. We shall refer to $\mathcal{W}$ as *the Weyl parameter*. $\mathcal{W}$ can be regarded as describing the intrinsic anisotropy in the gravitational field. Cosmological observations can in principle give an upper bound on $\mathcal{W}$, although obtaining a strong bound is beyond the reach of present day observations. We refer specifically to observations of the CMBR [6] (see Maartens et al 1995b), and observations of distant galaxies (see Kristian & Sachs 1966, page 398).

The generalized Friedmann equation

$$3H^2 = \sigma^2 - \tfrac{1}{2}\,{}^3R + \mu,\tag{2.5}$$

where ^3R is the scalar curvature of the spacelike hypersurfaces of homogeneity, places bound on Ω and Σ^2. Dividing (2.5) by $3H^2$ yields

$$1 = \Sigma^2 + K + \Omega,\tag{2.6}$$

where

$$K = -\frac{{}^3R}{6H^2}$$

is a dimensionless quantity. Since $K \geq 0$ for Bianchi type VII_0 geometries [7], equation (2.6) implies that

$$\Sigma^2 \leq 1 \quad \text{and} \quad \Omega \leq 1.\tag{2.7}$$

On the other hand there is no general upper bound for the Weyl parameter $\mathcal{W}$. Indeed we shall show that $\mathcal{W}$ *is unbounded in the state space of the Bianchi VII_0 cosmologies.*

We now describe in turn the asymptotic behaviour of the three primary expansion-normalized scalars Ω, Σ and $\mathcal{W}$ in Bianchi VII_0 universes at late times, i.e. as $\tau \to +\infty$. These results will follow immediately from Theorems 3.1 - 3.3 in Section 3.

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[6] The analysis of Maartens et al 1995b does lead to a strong upper bound on the Weyl parameter $\mathcal{W}$ (see their equation (41)). However, they make an assumption concerning the time derivatives of the temperature harmonics (assumption C2$'$) which is not observationally testable at present.

[7] This result can be established using equations (1.95), (1.103) and (1.105) in WE.

Theorem 2.2:
For any non-tilted perfect fluid Bianchi VII$_0$ universe that is not LRS, the density parameter Ω satisfies

$$\lim_{\tau \to +\infty} \Omega = \begin{cases} 1 & , \quad \text{if} \quad \frac{2}{3} < \gamma \le \frac{4}{3} \\ \frac{3}{2}(2 - \gamma), & \quad \text{if} \quad \frac{4}{3} < \gamma < 2. \end{cases}$$

Theorem 2.3:
For any non-tilted perfect fluid Bianchi VII$_0$ universe that is not LRS, the shear parameter Σ satisfies

$$\lim_{\tau \to +\infty} \Sigma^2 = 0, \quad \text{if} \quad \frac{2}{3} < \gamma \le \frac{4}{3}$$

and

$$\limsup_{\tau \to +\infty} \Sigma^2 = \tfrac{1}{2}(3\gamma - 4), \quad \liminf_{\tau \to +\infty} \Sigma^2 = \tfrac{1}{4}(3\gamma - 4)^2, \quad \text{if} \quad \tfrac{4}{3} < \gamma < 2.$$

Theorem 2.4:
For any non-tilted perfect fluid Bianchi VII$_0$ universe which is not LRS, the Weyl curvature parameter $\mathcal{W}$ satisfies

$$\lim_{\tau \to +\infty} \mathcal{W} = \begin{cases} 0 & , \quad \text{if} \quad \frac{2}{3} < \gamma < 1, \\ L \ne 0, & \quad \text{if} \quad \gamma = 1, \\ +\infty & , \quad \text{if} \quad 1 < \gamma < 2. \end{cases}$$

where L is a constant that depends on the initial conditions, and can have any positive real value.

The implications of Theorems 2.3 and 2.4 are as follows:

i) there is a *shear bifurcation* at $\gamma = \frac{4}{3}$; specifically, if $\gamma > \frac{4}{3}$ the models no longer isotropize as regards the shear of the Hubble flow, and in fact $\lim_{\tau \to +\infty} \Sigma^2$ does not exist since Σ^2 is oscillatory as $\tau \to +\infty$.

ii) there is a *Weyl curvature bifurcation* at $\gamma = 1$; specifically, if $\gamma = 1$, the models no longer isotropize as regards the Weyl curvature, and if $1 < \gamma < 2$ the Weyl curvature moreover dominates the rate of expansion at late times.

Comment: By Theorems 2.2 - 2.4, the limits as $\tau \to +\infty$ of Ω, Σ and $\mathcal{W}$ exist if $\frac{2}{3} < \gamma \le 1$. One might thus expect that the models are asymptotically self-similar in this case, contradicting Theorem 2.1. We will show in Section 3, however, that if $\frac{2}{3} < \gamma \le 1$, a dimensionless scalar representing a time derivative of the Weyl tensor is unbounded as $\tau \to +\infty$.

3 Evolution equations and asymptotic behaviour

In order to write the evolution equations for the non-tilted perfect fluid Bianchi universes of group type VII$_0$ in a suitable form, we use the orthonormal-frame formalism, introduced in cosmology by Ellis & MacCallum 1969, in which the commutation functions γ^c_{ab} of the orthonormal frame $\{\mathbf{e}_a\}$, defined by

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c_{ab}\mathbf{e}_c,$$

act as the basic variables of the gravitational field. We choose the frame to be invariant under the isometry group G$_3$, and aligned with the fluid velocity $\mathbf{u}$ in the sense that $\mathbf{e}_0 = \mathbf{u}$. It follows that the γ^c_{ab} depend only on t, the clock time along the fluid congruence. For Bianchi universes of group type VII$_0$ one can specialize the frame so that the only non-zero commutation functions are the Hubble scalar $H = \frac{1}{3}\Theta$, where Θ is the rate of expansion scalar, and the diagonal components of the shear tensor $\sigma_{\alpha\beta}$ and a matrix $n_{\alpha\beta}$,

$$\sigma_{\alpha\beta} = \mathrm{diag}(\sigma_{11}, \sigma_{22}, \sigma_{33}), \quad n_{\alpha\beta} = \mathrm{diag}(0, n_2, n_3), \tag{3.1}$$

with $n_2 > 0$ and $n_3 > 0$. (see Ellis & MacCallum 1969). Since $\sigma_{\alpha\beta}$ is trace-free, it has only two independent components, which it is convenient to label using

$$\sigma_+ = \tfrac{1}{2}(\sigma_{22} + \sigma_{33}), \quad \sigma_- = \tfrac{1}{2\sqrt{3}}(\sigma_{22} - \sigma_{33}). \tag{3.2}$$

In analogy we label the components n_2, n_3 by

$$n_+ = \tfrac{1}{2}(n_2 + n_3), \quad n_- = \tfrac{1}{2\sqrt{3}}(n_2 - n_3) \tag{3.3}$$

The physical state of the models is thus described by the variables

$$(H, \sigma_+, \sigma_-, n_+, n_-). \tag{3.4}$$

We now introduce expansion-normalized variables and a dimensionless time τ according to

$$\Sigma_\pm = \frac{\sigma_\pm}{H}, \quad N_\pm = \frac{n_\pm}{H}, \tag{3.5}$$

and replace t by the dimensionless time variable τ according to equation (1.4) (see WE, Chapter 5, for background and motivation). The evolution equation for H can now be written as

$$H' = -(1 + q)H, \tag{3.6}$$

where $'$ denotes differentiation with respect to τ, and q is the deceleration parameter, given by

$$q = 2\Sigma^2 + \tfrac{1}{2}(3\gamma - 2)\Omega. \tag{3.7}$$

(see WE, pages 113-4). In addition Σ^2 and Ω are given by

$$\Sigma^2 = \Sigma_+^2 + \Sigma_-^2, \tag{3.8}$$

$$\Omega = 1 - \Sigma_+^2 - \Sigma_-^2 - N_-^2. \tag{3.9}$$

The evolution equations for $\Sigma_\pm$ and $N_\pm$ are

$$\begin{aligned}
\Sigma_+' &= -(2-q)\Sigma_+ - 2N_-^2 \\
\Sigma_-' &= -(2-q)\Sigma_- - 2N_-N_+ \\
N_+' &= (q + 2\Sigma_+)N_+ + 6\Sigma_-N_- \\
N_-' &= (q + 2\Sigma_+)N_- + 2\Sigma_-N_+.
\end{aligned} \tag{3.10}$$

These equations follow from equations (6.9), (6.10) and (6.35) in WE on setting $N_1 = 0$. For future reference we also note the evolution equation for Ω:

$$\Omega' = [2q - (3\gamma - 2)]\Omega, \tag{3.11}$$

(see WE, page 115).

The dimensionless state vector is $(\Sigma_+, \Sigma_-, N_+, N_-)$ and the physical region of state space is defined by the condition $\Omega \geq 0$, i.e.

$$\Sigma_+^2 + \Sigma_-^2 + N_-^2 \leq 1, \tag{3.12}$$

and the restrictions

$$N_+^2 - 3N_-^2 > 0, \quad N_+ > 0, \tag{3.13}$$

which are a consequence of (3.3), (3.5) and the restrictions $n_2 > 0$, $n_3 > 0$. The variables Σ_+, Σ_- and N_- are thus bounded, but N_+ can take on any real value. Indeed, we show in Appendix A that if $\Omega > 0$ and $\frac{2}{3} < \gamma < 2$, then for any initial conditions,

$$\lim_{\tau \to +\infty} N_+ = +\infty. \tag{3.14}$$

The variable N_+ does not appear in the equation for Σ_+', but does play a significant role in the equations for Σ_-' and N_-', which are of the form

$$\begin{aligned}
\Sigma_-' &= -2N_-N_+ + \{bounded\,terms\} \\
N_-' &= 2\Sigma_-N_+ + \{bounded\,terms\}.
\end{aligned}$$

The leading terms generate oscillations in Σ_- and N_-, and suggest that we introduce polar coordinates in the Σ_-N_--space. We also replace the variable N_+ by $1/N_+$ so as to have a variable that remains bounded as $\tau \to +\infty$. We thus let

$$\Sigma_- = R\cos\psi, \quad N_- = R\sin\psi, \tag{3.15}$$

where $R \geq 0$, and

$$M = \frac{1}{N_+}. \tag{3.16}$$

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A straightforward calculation using (3.10), (3.15) and (3.16) yields the evolution equations for the new variables (Σ_+, R, M, ψ) in the following form:

$$\Sigma'_+ = -(2 - Q)\Sigma_+ - R^2 + (1 + \Sigma_+)R^2 \cos 2\psi \tag{3.17}$$

$$R' = \left[Q + \Sigma_+ - 1 + (R^2 - 1 - \Sigma_+)\cos 2\psi\right]R \tag{3.18}$$

$$M' = -\left[Q + 2\Sigma_+ + R^2(\cos 2\psi + 3M\sin 2\psi)\right]M \tag{3.19}$$

$$\psi' = \frac{1}{M}\left[2 + (1 + \Sigma_+)M\sin 2\psi\right], \tag{3.20}$$

where

$$Q = 2\Sigma_+^2 + R^2 + \tfrac{1}{2}(3\gamma - 2)\Omega, \tag{3.21}$$

$$\Omega = 1 - \Sigma_+^2 - R^2 \tag{3.22}$$

In writing these equations we have expressed the trigonometric dependence in terms of $\cos 2\psi$ and $\sin 2\psi$. In particular, we have decomposed the deceleration parameter q, as given by equation (3.7), into a non-oscillatory and an oscillatory part,

$$q = Q + R^2 \cos 2\psi. \tag{3.23}$$

The variables Σ_+, R, M and ψ are required to satisfy the following restrictions. First, the requirement $\Omega \geq 0$ is equivalent to

$$\Sigma_+^2 + R^2 \leq 1. \tag{3.24}$$

Second, the restrictions (3.13) in conjunction with the definitions (3.15) and (3.16) lead to

$$M > 0, \quad R \geq 0, \quad 3R^2 M^2 \sin^2 \psi \leq 1. \tag{3.25}$$

We note that if $R = 0$ equation (3.20) for ψ' becomes irrelevant, and the remaining evolution equations (3.17) and (3.19) describe the LRS Bianchi VII_0 models (which are equivalent to LRS Bianchi I models).

We now state the main results concerning the asymptotic behaviour of the solutions of the DE (3.17) - (3.20) with $R > 0$, $\quad \Omega > 0 \quad$ and $\quad \tfrac{2}{3} < \gamma < 2$. For convenience, we introduce a new parameter β given by

$$\beta = \tfrac{1}{2}(4 - 3\gamma), \tag{3.26}$$

and note the relations

$$\tfrac{3}{2}(2 - \gamma) = 1 + \beta, \quad \tfrac{1}{2}(3\gamma - 2) = 1 - \beta, \quad 3(\gamma - 1) = 1 - 2\beta. \tag{3.27}$$

Theorem 3.1:
If $\frac{2}{3} < \gamma < \frac{4}{3}$, any solution of the DE (3.17) - (3.20) with $R > 0$ and $\Omega > 0$ satisfies

$$\lim_{\tau \to +\infty} M = 0, \quad \lim_{\tau \to +\infty} R = 0, \quad \lim_{\tau \to +\infty} \Sigma_+ = 0, \qquad (3.28)$$

and

$$
\begin{aligned}
M &= C_M e^{-(1-\beta)\tau} \left[1 + O\!\left(e^{-b\tau}\right) \right] \\
R &= C_R e^{-\beta\tau} \left[1 + O\!\left(e^{-b\tau}\right) \right] \\
\Sigma_+ &= -\frac{C_R^2}{1 - \beta} e^{-2\beta\tau} \left[1 + O\!\left(e^{-b\tau}\right) \right],
\end{aligned}
\qquad (3.29)
$$

as $\tau \to +\infty$, where β is given by (3.26), C_R and C_M are constants that depend on the initial conditions, and b is a positive constant.

Theorem 3.2:
If $\gamma = \frac{4}{3}$, any solution of the DE (3.17) - (3.20) with $R > 0$ and $\Omega > 0$ satisfies (3.28), and

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\begin{aligned}
M &= C_M \tau e^{-\tau} \left[1 + O\!\left(\tfrac{ln\tau}{\tau}\right) \right] \\
R &= \tfrac{1}{\sqrt{2}} \tau^{-\frac{1}{2}} \left[1 + O\!\left(\tfrac{ln\tau}{\tau}\right) \right] \\
\Sigma_+ &= -\tfrac{1}{2} \tau^{-1} \left[1 + O\!\left(\tfrac{ln\tau}{\tau}\right) \right],
\end{aligned}
\qquad (3.30)
$$

where C_M is a constant that depends on the initial conditions.

Theorem 3.3:
If $\frac{4}{3} < \gamma < 2$, any solution of the DE (3.17) - (3.20) with $R > 0$ and $\Omega > 0$ satisfies

$$\lim_{\tau \to +\infty} M = 0 \quad , \quad \lim_{\tau \to +\infty} \Sigma_+ = \beta \quad , \quad \lim_{\tau \to +\infty} R^2 = -\beta(1 + \beta) \quad , \qquad (3.31)$$

with

$$M = C_M e^{-(1+\beta)\tau} \left[1 + O\!\left(e^{-b\tau}\right) \right], \qquad (3.32)$$

where β is given by (3.26), C_M is a constant that depends on the initial conditions and b is a positive constant.

Theorems 3.1 - 3.3 enable us to determine the asymptotic behaviour of any physical or geometrical quantity in a non-tilted Bianchi VII_0 universe, and in particular, provide the proofs for Theorems 2.2 - 2.4. The shear parameter Σ is obtained from equations (3.8) and (3.15), and the density parameter Ω from equation (3.22):

$$\Sigma^2 = \Sigma_+^2 + R^2 \cos^2\psi, \quad \Omega = 1 - \Sigma_+^2 - R^2. \qquad (3.33)$$

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Theorems 3.1-3.3, in conjunction with equation (3.33), now lead directly to Theorems 2.2 and 2.3.

Next, the asymptotic form of the Hubble parameter H is determined algebraically, since the density parameter $\Omega = \mu/(3H^2)$ has a non-zero limit as $\tau \to +\infty$, for $\frac{2}{3} < \gamma < 2$, and the matter density is given by $\mu = \mu_0(l/l_0)^{-3\gamma}$, or equivalently, $\mu = \mu_0 e^{-3\gamma\tau}$, as follows from the contracted Bianchi identities. Knowing H, the relation between clock time t and dimensionless time τ can be obtained by integrating equation (1.4). On introducing the parameter β, we obtain

$$H \approx H_0 e^{-(2-\beta)\tau}, \quad H_0 t \approx \frac{1}{2-\beta} e^{(2-\beta)\tau}, \tag{3.34}$$

as $\tau \to +\infty$. The asymptotic expansions in Theorems 3.1-3.3 also determine the asymptotic form of the line-element by a quadrature and by algebraic means (See Appendix C).

We now discuss the behaviour of the Weyl curvature. We require an explicit expression for the Weyl parameter $\mathcal{W}$, defined by equations (2.3) and (2.4). For non-tilted Bianchi VII$_0$ models, E_{ab} and H_{ab}, and their dimensionless counterparts

$$\mathcal{E}_{ab} = \frac{E_{ab}}{H^2} \quad , \quad \mathcal{H}_{ab} = \frac{H_{ab}}{H^2}, \tag{3.35}$$

are diagonal relative to the standard orthonormal frame (see WE, Chapter 6, Appendix). In analogy with (3.2) we define

$$\mathcal{E}_+ = \tfrac{1}{2}(\mathcal{E}_{22} + \mathcal{E}_{33}) \quad , \quad \mathcal{E}_- = \tfrac{1}{2\sqrt{3}}(\mathcal{E}_{22} - \mathcal{E}_{33})$$
$$\mathcal{H}_+ = \tfrac{1}{2}(\mathcal{H}_{22} + \mathcal{H}_{33}) \quad , \quad \mathcal{H}_- = \tfrac{1}{2\sqrt{3}}(\mathcal{H}_{22} - \mathcal{H}_{33}). \tag{3.36}$$

It then follows from equations (2.3) and (2.4) that

$$\mathcal{W}^2 = \mathcal{E}_+^2 + \mathcal{E}_-^2 + \mathcal{H}_+^2 + \mathcal{H}_-^2. \tag{3.37}$$

In terms of the variables Σ_+, R, M, ψ as defined by equations (3.15) and (3.16), the components $\mathcal{E}_\pm$ and $\mathcal{H}_\pm$ are given by

$$\mathcal{E}_+ = \Sigma_+(1 + \Sigma_+) + \tfrac{1}{2}R^2(1 - 3\cos2\psi)$$
$$\mathcal{H}_+ = -\tfrac{3}{2}R^2\sin2\psi$$
$$\mathcal{E}_- = \tfrac{2R}{M}\left[\sin\psi + \tfrac{1}{2}M(1 - 2\Sigma_+)\cos\psi\right] \tag{3.38}$$
$$\mathcal{H}_- = \tfrac{2R}{M}\left[-\cos\psi - \tfrac{3}{2}M\Sigma_+\sin\psi\right].$$

These expressions follow from equations (6.36) and (6.37) in WE. Since Σ_+ and R are bounded, it follows from (3.37) and (3.38) that

$$\mathcal{W} = \frac{2R}{M}[1 + O(M)], \quad \text{as} \quad \tau \to +\infty. \tag{3.39}$$

This result, in conjunction with Theorems 3.1-3.3, yields

$$\mathcal{W}^2 = \begin{cases} 4\left(\frac{C_R}{C_M}\right)^2 e^{2(1-2\beta)\tau}\left[1 + O(e^{-b\tau})\right], & \text{if} \quad \frac{2}{3} < \gamma < \frac{4}{3} \\ \frac{2}{C_M^2}\tau^{-3}e^{2\tau}\left[1 + O(\frac{ln\tau}{\tau})\right], & \text{if} \quad \gamma = \frac{4}{3} \\ \frac{-4\beta(1+\beta)}{C_M^2}e^{2(1+2\beta)\tau}\left[1 + O(e^{-b\tau})\right], & \text{if} \quad \frac{4}{3} < \gamma < 2. \end{cases} \tag{3.40}$$

Theorem 2.4 follows immediately from this result using equation (3.27). We note that $\mathcal{W}$ diverges most rapidly for $\gamma = \frac{4}{3}$.

As mentioned at the end of Section 2, if $\frac{2}{3} < \gamma \leq 1$ the limits of the primary dimensionless scalars Σ^2, Ω and $\mathcal{W}$ exist, even though the model is not asymptotically self-similar. We can use the asymptotic expressions in Theorem 3.1 to show that *if $\frac{2}{3} < \gamma \leq 1$, there is an expansion-normalized scalar that is unbounded as $\tau \to +\infty$*. We need to know the dominant term in the derivatives of $\mathcal{E}_-^{(n)}$ and $\mathcal{H}_-^{(n)}$, where (n) denotes the nth derivative with respect to τ. One can calculate $\mathcal{E}_-^{(1)}$ and $\mathcal{H}_-^{(1)}$ by differentiating equations (3.38) with respect to τ, using the evolution equations (3.17) - (3.20). The dominant term arises from differentiating the leading trigonometric function, and we obtain

$$\mathcal{E}_-^{(1)} = \frac{4R}{M^2}[\cos\psi + O(M)],$$
$$\mathcal{H}_-^{(1)} = \frac{4R}{M^2}[\sin\psi + O(M)], \tag{3.41}$$

as $\tau \to +\infty$. Repeating this process shows that the dominant term in $\mathcal{E}_-^{(n)}$ and $\mathcal{H}_-^{(n)}$ grows like R/M^{n+1} as $\tau \to +\infty$. Theorem 3.1 implies that

$$\frac{R}{M^{n+1}} \sim e^{[n(1-\beta)-(2\beta-1)]\tau} \quad \text{as} \quad \tau \to +\infty.$$

It follows from this result that $\mathcal{E}_-^{(n)}$ and $\mathcal{H}_-^{(n)}$ are unbounded as $\tau \to +\infty$ if and only if $n > 6(1-\gamma)/(3\gamma-2)$. Thus, for any γ in the range $\frac{2}{3} < \gamma \leq 1$, $\mathcal{E}_-^{(n)}$ and $\mathcal{H}_-^{(n)}$ are unbounded as $\tau \to +\infty$ for n sufficiently large.

We conclude this Section by giving a heuristic justification of Theorems 3.1 - 3.3. The key result underlying the proof is that

$$\lim_{\tau \to +\infty} M = 0, \tag{3.42}$$

as follows from equations (3.14) and (3.16). We shall show that in fact M tends to zero at an exponential rate. Equation (3.20) implies that once M is close to zero, ψ' is large, and that ψ grows exponentially as $\tau \to +\infty$. Because of this exponential growth, *the solutions of the evolution equations (3.17) - (3.20) are approximated by solutions of the simplified DE obtained by dropping the $\cos 2\psi$ and $\sin 2\psi$ terms in equations (3.17) and (3.18):*

$$\hat{\Sigma}'_+ = -(2 - \hat{Q})\hat{\Sigma}_+ - \hat{R}^2 \tag{3.43}$$
$$\hat{R}' = (\hat{Q} + \hat{\Sigma}_+ - 1)\hat{R},$$

where we use the symbol ˆ to distinguish the variables in (3.43) from those in equations (3.17), (3.18) and (3.21). This behaviour, which is shown very convincingly in numerical

experiments, is due to the fact that the $\cos 2\psi$ and $\sin 2\psi$ terms are essentially nullified since they change sign repeatedly over an increasingly short time scale.

The state space for the variables $\hat{\Sigma}_+$ and $\hat{R}$ is

$$\hat{\Sigma}_+^2 + \hat{R}^2 \leq 1, \quad \hat{R} \geq 0$$

(see (3.24) and (3.25)). Since this set is closed and bounded we can use standard methods to analyze the asymptotic behaviour of solutions of the DE (3.43). We find that if $\frac{2}{3} < \gamma \leq \frac{4}{3}$ the point $(\hat{\Sigma}_+, \hat{R}) = (0,0)$ is a global sink for solutions with $\Omega > 0$, and if $\frac{4}{3} < \gamma < 2$, the point $(\hat{\Sigma}_+, \hat{R}) = (\beta, \sqrt{-\beta(1+\beta)})$ is a global sink for solutions with $\Omega > 0$ and $R > 0$. One can also determine the asymptotic form of the solutions of the simplified DE (3.43), which gives precisely the leading asymptotic dependence on τ in Theorem 3.1 - 3.3.

In summary, for any initial conditions, equation (3.42) implies that M will be close to zero for τ sufficiently large, after which the evolution will be governed asymptotically by the simplified DE (3.43). Proving the theorems thus hinges on proving that the oscillatory terms do in fact become negligible for τ sufficiently large. A detailed proof of Theorem 3.1 is given in Appendix B. The proof of Theorem 3.3 (the case $\frac{4}{3} < \gamma < 2$) is of a similar nature to that of Theorem 3.1, while the proof of Theorem 3.2 (the case of $\gamma = \frac{4}{3}$) is more complicated, since the global sink for the averaged DE (3.43) is non-hyperbolic. The proof thus requires the use of cente manifold theory (see for example Carr 1981). The details of this case will be published elsewhere.

4   Discussion

In this paper we have described for the first time the evolution into the future of any non-tilted Bianchi VII$_0$ universe with perfect fluid matter content and equation of state $p = (\gamma - 1)\mu$, for γ in the range $\frac{2}{3} < \gamma < 2$. We have also given asymptotic forms as $\tau \to +\infty$ of various physical and geometrical quantities of interest. Although certain aspects of this problem have been studied before, most notably by [8] Collins & Hawking 1973a, Doroshkevich et al 1973 and Barrow & Sonoda 1986, a general treatment has not been given, and moreover a number of important properties of the dynamics have remained unnoticed.

The most significant feature of the late time evolution is the breaking of asymptotic self-similarity (Theorem 2.1), characterized by oscillations that become increasingly rapid in terms of dimensionless time τ as $\tau \to +\infty$. This oscillatory behaviour leads to the phenomenon of *Weyl curvature dominance*, i.e. expansion-normalized scalars constructed from the Weyl tensor become unbounded as $\tau \to +\infty$. We note that this type of asymptotic symmetry breaking is quite different from that displayed by the Mixmaster models in the singular asymptotic regime, which has a stochastic nature but with all expansion-normalized scalars remaining bounded.

It is worth noting that both types of asymptotic self-similarity breaking create difficulties when one attempts to solve the Einstein field equations numerically. In the Bianchi VII$_0$ case, if one uses τ as the time variable the increasingly rapid oscillations (like $\cos(e^{\lambda\tau})$) cause

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[8]We discuss the results of these authors in Appendix C, when we describe the asymptotic form of the line-element.

numerical difficulties, while if one changes to a time variable relative to which the oscillations are asymptotically periodic (e.g. conformal time η; see table C.1) then the overall evolution slows to such an extent that one cannot integrate very far into the late time asymptotic regime. On the other hand, as one follows the evolution of a Mixmaster model into the past, the model spends an increasingly long time (τ-time) in the successive Kasner states so that it is likewise difficult to integrate far into the singular asymptotic regime. [9]

The late time behaviour of the Bianchi VII_0 models also has interesting implications as regards the question of isotropization. In a spatially homogeneous cosmology, there are two physical manifestations of anisotropy,

i) the *shear* of the timelike congruence that represents the large scale distribution of the matter,

ii) the *Weyl curvature*, which can be viewed as describing the intrinsic anisotropy in the gravitational field: it determines up to four preferred directions, the principal null directions of the gravitational field.

The shear parameter Σ and the Weyl parameter $\mathcal{W}$ quantify these anisotropies.

In discussions of isotropization in the literature restrictions are imposed on Σ, namely

$$\lim_{\tau \to +\infty} \Sigma = 0, \tag{4.1}$$

corresponding to *asymptotic isotropization* (see for example, Collins and Hawking 1973b page 324), or

$$\Sigma \ll 1, \tag{4.2}$$

over some finite time interval [10], corresponding to *intermediate isotropization* (see Zeldovich and Novikov 1983, page 550, and more recently, Wainwright et al 1998, pages 331-2 and 343-4). Restrictions are not usually imposed on the Weyl curvature. For the class of non-tilted Bianchi models $\Sigma = 0$ characterizes the FL models and hence implies that $\mathcal{W} = 0$, making it tempting to conjecture that (4.1) and (4.2) imply the corresponding results for the Weyl parameter $\mathcal{W}$. Our analysis shows decisively that this conjecture is false: even though (4.1) holds, the Weyl parameter $\mathcal{W}$ can be arbitrarily large for dust ($\gamma = 1$) and is unbounded for the parameter range $1 < \gamma \le \frac{4}{3}$ (which includes radiation), as $\tau \to +\infty$. The implication is that *a model which satisfies* (4.1) *or* (4.2) *is not necessarily close to isotropy, since $\mathcal{W}$ may not be small.*

It is of interest to compare the late time evolution of Bianchi VII_0 universes with the evolution of perturbations of the flat FL model. We can do this by linearizing the Bianchi VII_0 evolution equations about the FL model, given by $\Sigma_+ = 0 = R$. Since the perturbed solutions are spatially homogeneous and have zero vorticity, it follows that the scalar and vector perturbations are zero, giving purely tensor perturbations (Goode 1989, Section V).

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[9] We refer to Berger et al 1997 for a new algorithm for improving the situation somewhat.

[10] For example, the time elapsed since last scattering or the time elapsed since nucleosynthesis.

The exact solutions of the linearized equations involve Bessel functions but for our purposes it is sufficient to know the asymptotic forms,

$$\Sigma_+^{lin} \approx \quad C_\Sigma e^{-(1+\beta)\tau} \quad , \quad R_{lin} \approx \quad C_R e^{-\beta\tau} \quad , \quad M_{lin} \approx \quad C_M e^{-(1-\beta)\tau}. \qquad (4.3)$$

Comparing this result with Theorem 3.1, we see that *the linearized equations predict the correct asymptotic form of R and M, for γ in the range* $\frac{2}{3} < \gamma < \frac{4}{3}$. This implies that solutions of the linearized equations lead to the same asymptotic form for the Weyl parameter $\mathcal{W}$ as the exact equations (see equation (3.39)). Thus, *the Weyl curvature bifurcation at $\gamma = 1$ is a property of the linearized solutions as well as the exact solutions.*

On the other hand, it follows from equation (4.3) and Theorem 3.1 that the linearized equations predict that Σ_+ decays faster than it actually does, i.e. the asymptotic form of Σ_+ is a non-linear effect. Furthermore, comparing equations (4.3) with Theorem 3.2, we see that the asymptotic form of all variables in the case of radiation ($\gamma = \frac{4}{3}$) is a non-linear phenomenon.

We now discuss the implications of our results as regards the temperature of the CMBR. We have seen that the shear of the Hubble flow, which is described by the two frame components $\Sigma_\pm$, has two noteworthy features, in particular for the case of dust ($\gamma = 1$) :

  i) the Σ_- component oscillates increasingly rapidly while tending to zero, as the dimensionless time $\tau \to +\infty$,

  ii) the rate of decay of the Σ_+ component is a non-linear effect.

The temperature of the CMBR is given by the formula [11]

$$T_o = T_e exp\left[-\int_{\tau_e}^{\tau_o}(1 + \Sigma_{\alpha\beta}K^\alpha K^\beta)d\tau\right], \qquad (4.4)$$

where the K^α are the direction cosines of a particular null geodesic, $\Sigma_{\alpha\beta} = \sigma_{\alpha\beta}/H$ is the dimensionless shear tensor, and τ is the dimensionless time. The subscripts o and e refer to the present time and the time of emission, respectively. It is thus evident that the link between the temperature anistotropy and the shear is complicated, particularly in view of the oscillatory nature of the shear, and that determining a reliable bound on the shear scalar Σ will necessitate integrating the full evolution equations and null geodesic equations numerically [12]. Without investigating this link further, we can, however, use the fact that small Σ does not imply small $\mathcal{W}$ to make the following assertion: *a highly isotropic CMBR temperature at the present time does not lead to restrictions on the Weyl parameter.* Indeed *the Bianchi VII_0 universes provide an example of cosmological models in which the CMBR temperature is highly isotropic but the Weyl parameter is not small.* As mentioned in Section 3, the analysis of Maartens et al 1995b shows that in order to obtain a bound on $\mathcal{W}$ one

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[11]In papers analysing the anisotropy of the temperature of the CMBR the linearized version of this formula, valid for $\Sigma\alpha\beta << 1$, is usually given. See for example Collins & Hawking 1973a.

[12]The anisotropy in the temperature of the CMBR in Bianchi VII_0 models has been investigated by Collins and Hawking 1973a and by Doroshkevich et al 1975. In contrast to Collins and Hawking, Doroshkevich et al argue that it is not sufficient to use linear perturbations of the FL models.

requires an observational bound on the *time derivatives* of the temperature harmonics, which is not attainable in practice.

We conclude with some remarks interpreting our results from a broader perspective. Despite the special nature of the models under consideration - they are spatially homogoneous and moreover are not of the most general Bianchi type - the complicated dynamical behaviour that we have described may have broad significance, for the following reason. The four-dimensional state space of the non-tilted Bianchi VII$_0$ models is an invariant subset of the state space for more general models, and by continuity the orbits in an open subset of the full state space will "shadow", i.e. approximate, the orbits in the invariant subset. In other words, *an open subset of more general classes of models will display the features of Bianchi VII$_0$ models*, at least during a finite time interval. Thus, for example, we can assert that there exist Bianchi VII$_0$ models with tilt, and Bianchi VIII or IX models with or without tilt, with the property that during some extended epoch the shear parameter Σ is small but the Weyl curvature parameter $\mathcal{W}$ is large. More generally, this property will also be displayed by spatially inhomogeneous cosmologies, for example G_2 cosmologies [13], which contain Bianchi VII$_0$ cosmologies as a special case. Based on our experience with the hierarchy of Bianchi models, namely that dynamical complexity increases with the dimension of the state space, we expect the dynamics of Bianchi models to provide a lower bound, so to speak, for the dynamical complexity of spatially inhomogeneous models. Understanding the dynamics of Bianchi universes is thus a necessary first step in the study of more general models.

Appendix A: Proof of Theorem 2.1

We consider the state space S of the Bianchi VII$_0$ models defined by the inequalities

$$N_+ > 0, \quad N_+^2 - 3N_-^2 > 0, \quad \Omega > 0, \tag{A.1}$$

(see (3.12) and (3.13)). The function

$$Z = \frac{(N_+^2 - 3N_-^2)^v \, \Omega}{(1 + v\Sigma_+)^{2(1+v)}}, \quad \text{with} \quad v = \tfrac{1}{4}(3\gamma - 2), \tag{A.2}$$

satisfies $0 < Z < +\infty$ on S, and the evolution equations (3.10) - (3.11) imply that

$$\frac{Z'}{Z} = \frac{4[(\Sigma_+ + v)^2 + (1 - v^2)\Sigma_-^2]}{1 + v\Sigma_+}. \tag{A.3}$$

Thus if $\frac{2}{3} \le \gamma \le 2$, Z is increasing along orbits in S, since $0 \le v \le 1$.

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[13]As an example of the dynamics of a special class of models occurring in a more general class (but for the singular asymptotic regime), we note that Weaver et al 1998 have recently shown numerically that Mixmaster-like oscillations do in fact occur in G_2-cosmologies in the presence of a magnetic field, generalizing the behaviour of the magnetic Bianchi VI$_0$ cosmologies, whose state space is an invariant subset of the infinite dimensional state space of the G_2-cosmologies (see Leblanc et al 1995)

The set of boundary points of S that are not contained in S is the set $\overline{S}/S$, where $\overline{S}$ is the closure of S. It follows from (A.1) that $\overline{S}/S$ is defined by one or both of the following equalities holding:

$$\Omega = 0 \quad , \quad N_+^2 - 3N_-^2 = 0.$$

By equation (A.2), Z is defined and equal to zero on $\overline{S}/S$. We can now apply the Monotonicity Principle (see WE, Theorem 4.12, with "decreasing" replaced by "increasing") to conclude that for any point $x \in S$, the ω-limit set $\omega(x)$ is contained in the subset of $\overline{S}/S$ that satisfies the condition $\lim_{y \to s} Z(y) \neq 0$, where $s \in \overline{S}/S$ and $y \in S$. This subset is the empty set, since $Z = 0$ on $\overline{S}/S$, we conclude that $\omega(x) = \phi$ for all $x \in S$.

We now prove that

$$\lim_{\tau \to +\infty} N_+ = +\infty \tag{A.4}$$

for each orbit in S. Suppose that (A.4) does not hold. Then there exists a number $b > 0$ such that for any τ_0, there exists a $\tau > \tau_0$ with $N_+(\tau) < b$. Since the other variables are bounded, this implies that the orbit $x(\tau)$ has infinitely many points in a compact subset of $S \subset \mathbb{R}^4$ and hence has a limit point in S, contradicting $\omega x) = \phi$. Thus (A.4) holds. In conclusion, we have proved that no orbit in S is future asymptotic to an equilibrium point in S and that for every orbit in S, (A.4) holds.

Appendix B: Proof of Theorem 3.1

The proof of Theorems 3.1 - 3.3 involves two main steps:

i) Prove that

$$\lim_{\tau \to +\infty} (\Sigma_+, R) = \begin{cases} (0,0), & \frac{2}{3} < \gamma \leq \frac{4}{3} \\ (\beta, \sqrt{-\beta(1+\beta)}), & \frac{4}{3} < \gamma < 2 \end{cases}$$

for all initial states.

ii) Deduce the asymptotic form of (Σ_+, R, M) as $\tau \to +\infty$.

The evolution equations (3.17) - (3.20) are of the form

$$\begin{aligned} x' &= f_0(x) + f_1(x) \cos 2\psi \\ M' &= \left[m_0(x) + m_1(x) \cos 2\psi + m_2(x) M \sin 2\psi \right] M \\ \psi' &= \frac{2}{M} \left[1 + s(x) M \sin \psi \right], \end{aligned} \tag{B.1}$$

where the arbitrary functions are polynomials in $x = (\Sigma_+, R)$. We recall equation (3.42), namely, $\lim_{\tau \to +\infty} M = 0$.

In the analysis we will encounter scalar functions

$$Z = Z(x, M),$$

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whose evolution equation is of the form

$$Z' = \big[z_0(x) + z_1(x)\cos 2\psi + z_2(x)M\sin 2\psi\big]Z, \tag{B.2}$$

where z_0, z_1 and z_2 are polynomials, and

$$z_0(0) = -\eta < 0, \tag{B.3}$$

It is understood that x, M and ψ in (B.2) satisfy the DE (B.1).

Suppose it is known that all solutions of the DE (B.1) satisfy

$$H_1: \quad \lim_{\tau \to +\infty} x(\tau) = 0,$$

or the stronger condition

$$H_2: \quad x(\tau) = O(e^{-b\tau}), \quad M = O(e^{-b\tau}) \quad \text{as} \quad \tau \to +\infty,$$

$$\text{for some constant} \quad b > 0.$$

These conditions in fact imply that Z tends to zero at an exponential rate. The desired conclusions, given in terms of the constant η in (B.3), are as follows,

$$C_1: \quad Z = O\big(e^{(-\eta+\delta)\tau}\big) \quad \text{as} \quad \tau \to +\infty, \text{for any} \quad \delta > 0,$$

$$C_2: \quad Z = C_z e^{-\eta\tau}\big[1 + O(e^{-b\tau})\big], \quad \text{as} \quad \tau \to +\infty,$$

$$\text{where } C_z \text{ is a constant that depends on the initial condition.}$$

Proposition B.1:

$$H_1 \quad \text{implies} \quad C_1 \quad \text{and} \quad H_2 \quad \text{implies} \quad C_2.$$

Proof: The idea is to rescale Z so as to "suppress" the rapidly oscillating term $z_1(x)\cos 2\psi$ in (B.2) which may not tend to zero as $\tau \to +\infty$.
Let

$$\overline{Z} = \frac{Z}{1 + \frac{1}{4}Mz_1(x)\sin 2\psi} \tag{B.4}$$

Differentiating (B.4) with respect to τ, and using (B.1) and (B.2) leads to an equation of the form

$$\overline{Z}' = [z_0(x) + MB]\overline{Z}, \tag{B.5}$$

where $B = B(x, M, \cos 2\psi, \sin 2\psi)$ is bounded for sufficiently large τ.

Suppose that H_1 holds. Since M tends to 0 as $\tau \to +\infty$, it follows from (B.3) that given $\delta > 0$,

$$z_0(x) + MB \le -\eta + \delta,$$

for all τ sufficiently large, leading to C_1.

Suppose that H_2 holds. Then (B.5) can be written in the form

$$\overline{Z}' = \left[-\eta + O(e^{-b\tau})\right]\overline{Z}$$

as $\tau \to +\infty$, and C_2 follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We shall also encounter evolution equations of the form

$$Z' = -\eta Z + B, \qquad\qquad\qquad\qquad\qquad\text{(B.6)}$$

where η is a constant and $B = B(x, M, \cos 2\psi, \sin 2\psi)$ is bounded. By multiplying by $e^{-\eta(\tau-\tau_0)}$ and integrating from τ_0 to τ, this DE can be written in the equivalent integral form

$$Z(\tau) = e^{-\eta(\tau-\tau_0)}Z(\tau_0) + \int_{\tau_0}^{\tau} e^{-\eta(\tau-s)}B\,ds, \qquad\qquad\text{(B.7)}$$

where the arguments of B are evaluated at s. In connection with this integral form, we will need

Gronwall's Lemma:
If $v(\tau)$ is a non-negative C^1 function, and

$$v(\tau) \le C + \delta \int_{\tau_0}^{\tau} v(s)ds,$$

where C, δ are positive constants, then

$$v(t) \le Ce^{\delta(\tau-\tau_0)} \quad \text{for all} \quad \tau \ge \tau_0.$$

Proof: See for example, Hirsch & Smale (1974), page 169.

We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1:
We first prove that all solutions of the DE (C.1) satisfy $\lim\limits_{\tau\to+\infty} x(\tau) = 0$. We write the evolution equation (3.11) for Ω in the form

$$\Omega' = 2\left[(1+\beta)\Sigma_+^2 + \beta R^2 + R^2\cos 2\psi\right]\Omega,$$

using (3.21) - (3.23) and (3.27).
In analogy with (B.4), we let

$$\overline{\Omega} = \frac{\Omega}{1 + \frac{1}{2}MR^2\sin 2\psi}. \qquad\qquad\qquad\qquad\text{(B.8)}$$

Differentiating with respect to τ and using (3.18) - (3.20) leads to an equation of the form

$$\overline{\Omega}' = 2\left[(1+\beta)\Sigma_+^2 + R^2(\beta + MB)\right]\overline{\Omega}, \qquad\qquad\text{(B.9)}$$

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where B is a bounded expression in $\Sigma_+, R, M, \cos 2\psi$ and $\sin 2\psi$, for τ sufficiently large. Since $\lim_{\tau \to +\infty} M = 0$ and $\beta > 0$, it follows from (B.9) that there exists τ_0 such that $\overline{\Omega}' \geq 0$ for all $\tau > \tau_0$. Since $\Omega \leq 1$, equation (B.8) and the bound (3.25) imply that $\overline{\Omega}$ is bounded above. It thus follows that $\lim_{\tau \to +\infty} \overline{\Omega}$ exists, say

$$\lim_{\tau \to +\infty} \overline{\Omega} = L, \tag{B.10}$$

which, by equations (B.8), and (3.42) gives $\lim_{\tau \to +\infty} \Omega = L$. Since $\Omega \leq 1$, we have $L \leq 1$. Suppose $L < 1$. Then for τ sufficiently large, it follows that $\Sigma_+^2 + R^2 = 1 - \Omega > \frac{1}{2}(1 - L)$, Equation (B.9) now implies

$$\frac{\overline{\Omega}'}{\overline{\Omega}} > \beta(1 - L) + 2\Sigma_+^2 + R^2 M B.$$

Since $\lim_{\tau \to +\infty} M = 0$ we have $|R^2 M B| < \frac{\beta}{2}(1 - L)$ for τ sufficiently large. It follows that

$$\frac{\overline{\Omega}'}{\overline{\Omega}} > \tfrac{1}{2}\beta(1 - L) > 0$$

for τ sufficiently large, contradicting (B.10). Thus $L = 1$, i.e. $\lim_{\tau \to +\infty} \Omega = 1$, which implies

$$\lim_{\tau \to +\infty} (\Sigma_+^2 + R^2) = 0, \quad \text{i.e.} \quad \lim_{\tau \to +\infty} x(\tau) = 0.$$

We can now use Proposition B.1, with H_1 satisfied, to show that R and M decay exponentially to zero. The evolution equations (3.18) and (3.19) are of the form (B.2) with the constant η in (B.3) given by

$$\eta = -2\beta \quad \text{for} \quad R^2, \quad \eta = -(1 - \beta) \quad \text{for} \quad M.$$

The proposition thus implies that

$$R^2 = O\big(e^{(-2\beta + \delta)\tau}\big), \quad M = O\big(e^{(-(1-\beta)+\delta)\tau}\big), \tag{B.11}$$

as $\tau \to +\infty$.

The evolution equation (3.17) for Σ_+ can be written in the form

$$\Sigma_+' = -(1 + \beta)\Omega\Sigma_+ - R^2(1 + \Sigma_+)(1 - \cos 2\psi) \tag{B.12}$$

using (3.21), (3.22) and (3.26). Further rearrangement yields

$$\Sigma_+' = -(1 + \beta)\Sigma_+ + B(x, \cos 2\psi),$$

where

$$|B| \quad \leq \quad (1 + \beta)|\Sigma_+|^3 + c_1 R^2, \tag{B.13}$$

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with c_1 a constant. By (B.7), this DE has the equivalent integral form

$$\Sigma_+(\tau) = e^{-(1+\beta)(\tau-\tau_0)}\Sigma_+(\tau_0) + \int_{\tau_0}^{\tau} e^{-(1+\beta)(\tau-s)} B ds \qquad (B.14)$$

where the arguments of B are evaluated at s. By equation (B.11),

$$R^2 < c_2 e^{(-2\beta+\delta)\tau} \quad \text{for} \quad \tau > \tau_0.$$

Since $\lim_{\tau\to+\infty} \Sigma_+ = 0$ by H_1, we can choose τ_0 large enough that

$$\Sigma_+^2 < \frac{\delta}{1+\beta} \quad \text{for} \quad \tau > \tau_0.$$

Using these inequalities with (B.13), it follows from equation (B.14) that [14]

$$|\Sigma_+(\tau)| \quad \leq \quad Ce^{(-2\beta+\delta)\tau} \quad + \quad \delta \int_{\tau_0}^{\tau} e^{(-2\beta+\delta)(\tau-s)} |\Sigma_+(s)| ds,$$

Multiplying by $e^{(2\beta-\delta)\tau}$ yields

$$v(\tau) \quad \leq \quad C + \delta \int_{\tau_0}^{\tau} v(s) ds,$$

where $v(\tau) = e^{(2\beta-\delta)\tau}|\Sigma_+(\tau)|$. Gronwall's lemma thus implies that

$$|\Sigma_+(\tau)| \quad \leq \quad Ce^{(-2\beta+2\delta)\tau} \quad \text{for} \quad \tau \geq \tau_0.$$

At this stage we have shown that hypothesis H_2 is satisfied. We can thus use Proposition (B.1) to conclude that

$$R = C_R e^{-\beta\tau}[1 + 0(e^{-\beta\tau})], \quad M = C_M e^{-(1-\beta)\tau}[1 + 0(e^{-b\tau})], \qquad (B.15)$$

as $\tau \to +\infty$ (see the discussion leading to equation (B.11).

The final step is to deduce the asymptotic form of $\Sigma_+(\tau)$. It is necessary to do a change of variable in order to suppress the rapidly oscillating term $\cos 2\psi$ in (B.12). Define

$$\overline{\Sigma}_+ = \Sigma_+ - \tfrac{1}{4}MR^2(1+\Sigma_+)\sin 2\psi. \qquad (B.16)$$

Differentiating (B.16) with respect to τ, and using (B.1) and (B.2) leads to an equation of the form

$$\Sigma'_+ = -(1+\beta)\overline{\Sigma}_+ - R^2 + \overline{B}, \qquad (B.17)$$

where

$$\overline{B} = (1+\beta)\Sigma_+^3 + \beta\Sigma_+ R^2 + MR^2 B,$$

---

[14]The inequality $-(1+\beta) < -2\beta + \delta$ is used to make all constants in the exponentials the same.

and B is a complicated expression that is bounded as $\tau \to +\infty$. By (B.7), the DE (B.17) has the equivalent integral form

$$\overline{\Sigma}_+(\tau) = e^{-(1+\beta)(\tau-\tau_0)}\Sigma_+(\tau_0) - \int_{\tau_0}^{\tau} e^{-(1+\beta)(\tau-s)}R^2 ds - \int_{\tau_0}^{\tau} e^{-(1+\beta)(\tau-s)}\overline{B}ds. \tag{B.18}$$

The rates of decay for Σ_+, R and M obtained so far imply that the integral involving R^2 is the dominant term as $\tau \to +\infty$. Using (B.15), this integral can be evaluated, giving

$$\int_{\tau_0}^{\tau} e^{-(1+\beta)(\tau-s)}R^2 ds = \frac{C_R^2}{1-\beta}e^{-2\beta\tau}\big[1 + O(e^{-\bar{b}\tau})\big], \tag{B.19}$$

as $\tau \to +\infty$, for some positive constant $\bar{b}$. The asymptotic form for $\Sigma_+(\tau)$,

$$\Sigma_+(\tau) = -\frac{C_R^2}{1-\beta}e^{-2\beta\tau}\big[1 + O(e^{-b\tau})\big],$$

as $\tau \to +\infty$, follows from (B.16), (B.18) and (B.19), on noting that the term MR^2 in (B.17) is of order $e^{-(1+\beta)\tau}$ as $\tau \to +\infty$. The value of the positive constant b in the error term is re-defined as necessary.

Appendix C: The metric approach

Other investigations of Bianchi VII_0 universes have used metric tensor components as basic variables. We can obtain the asymptotic behaviour of the metric components directly from our results in Section 3, and we now do this for the purpose of comparison.

For any Bianchi model one can introduce a set of group-invariant and time-independent one-forms $W^\alpha, \alpha = 1, 2, 3$, relative to which the line-element has the form

$$ds^2 = -dt^2 + g_{\alpha\beta}(t)W^\alpha W^\beta, \tag{C.1}$$

where t is clock time along the normal congruence of the group orbits (e.g. WE, page 39). For group type VII_0, the one-forms can be chosen to satisfy

$$dW^1 = 0, \quad dW^2 = W^3 \wedge W^1, \quad dW^3 = W^1 \wedge W^2. \tag{C.2}$$

If the matter-energy content is a non-tilted perfect fluid, then $g_{\alpha\beta}(t)$ can be diagonalized, and for our purposes it is convenient to use the Misner labelling of the metric components.

$$ds^2 = -dt^2 + l^2\big[e^{-4\beta^+}(W^1)^2 + e^{2(\beta^+ + \sqrt{3}\beta^-)}(W^2)^2 + e^{2(\beta^+ - \sqrt{3}\beta^-)}(W^3)^2\big] \tag{C.3}$$

(Misner 1969, page 1323), where l is the length scale function. The structure relations are invariant under a rescaling [15] $W^2 \to \lambda W^2, \quad W^3 \to \lambda W^3$, where λ is a constant, which implies that there is freedom in redefining l and β^+ according to

$$l \to e^{2C}l, \quad \beta^+ \to \beta^+ + C, \tag{C.4}$$

---

[15]This transformation is an element of the automorphism group for Bianchi VII_0 (see, for example, Jantzen 1984).

where C is a constant.

The relations between the metric variables and the orthonormal frame variables are given in WE (Chapter 9), for Bianchi models of class A. Specializing equations (10.5) and (10.44) in WE to Bianchi VII$_0$ and using (10.7), (10.11), (6.8) and (6.35) in WE leads to

$$N_+ + \sqrt{3}N_- = \frac{1}{Hl}e^{2(\beta^+ + \sqrt{3}\beta^-)}, \quad N_+ - \sqrt{3}N_- = \frac{1}{Hl}e^{2(\beta^+ - \sqrt{3}\beta^-)}. \tag{C.5}$$

Introducing the variables R, M and ψ by equations (3.15) and (3.16) gives

$$\tanh(2\sqrt{3}\beta^-) = \sqrt{3}RM\sin\psi, \tag{C.6}$$

$$e^{4\beta^+} = \frac{l^2 H^2}{M^2}(1 - 3R^2 M^2 \sin^2\psi). \tag{C.7}$$

In addition, WE (10.46) reads

$$\Sigma_\pm = \frac{d\beta^\pm}{d\tau}, \tag{C.8}$$

where τ is the dimensionless time variable defined by equation (1.4). We note that the length scale l is related to τ by equation (1.3)

We can now use the asymptotic expressions for Σ_+, R, M and ψ in Theorems 3.1 - 3.3 to find the asymptotic form of β^+ and β^-, and the relation between clock time t and dimensionless time τ. Since $M \to 0$ as $\tau \to +\infty$, equation (C.6) implies that $\beta^- \to 0$ as $\tau \to 0$, and that for $|\beta^-| \ll 1$ we have $\beta^- \approx \frac{1}{2}RM\sin\psi$, which determines β^- directly. We integrate (C.8) to find β^+ up to an additive constant.

By using the freedom (C.4) in conjunction with equations (1.3), (3.34) and (C.7) we can arrange that the constant C_M in the asymptotic form of M in Section 3 is related to H_0 and l_0 by

$$C_M = H_0 l_0. \tag{C.9}$$

We now give the asymptotic expressions for β^+ and β^- as $\tau \to +\infty$ in the three cases.

i) $\frac{2}{3} < \gamma < \frac{4}{3}$:

$$\beta^+ \approx \frac{C_R^2}{2\beta(1-\beta)}e^{-2\beta\tau}, \quad \beta^- \approx \frac{1}{2}C_R H_0 l_0 e^{-\tau}\sin[\psi(\tau)], \tag{C.10}$$

     with

$$\psi(\tau) \approx \frac{2}{H_0 l_0 (1-\beta)}e^{(1-\beta)\tau} + \psi_0$$

ii) $\gamma = \frac{4}{3}$:

$$\beta^+ \approx -\frac{1}{2}ln\tau, \quad \beta^- \approx \frac{1}{2\sqrt{2}}H_0 l_0 \tau^{\frac{1}{2}}e^{-\tau}\sin[\psi(\tau)], \tag{C.11}$$

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with

$$\psi'(\tau) \approx \frac{2}{H_0 l_0} \frac{e^\tau}{\tau}.$$

In this case the asymptotic form of $\psi(\tau)$ cannot be expressed in terms of elementary functions.

iii) $\quad \frac{4}{3} < \gamma < 2 :$

$$\beta^+ \approx -\beta\tau, \quad \beta^- \approx \sqrt{-\beta(1+\beta)}e^{-(1+\beta)\tau} \sin\left[\psi(\tau)\right], \qquad (C.12)$$

with

$$\psi(\tau) \approx \frac{2}{H_0 l_0 (1+\beta)} e^{(1+\beta)\tau} + \psi_0.$$

We now briefly describe the results concerning the late-time asymptotic behaviour of Bianchi VII$_0$ universes that have been given in the literature. Comparing the results is made difficult by the fact that a variety of different time variables have been used. For convenience we summarize the time variables in Table C.1. (see also WE, page 244). The asymptotic dependence follows from

Name	Definition	Values	Dependence on τ as $\tau \to +\infty$
Clock time t	$\frac{dt}{d\tau} = \frac{1}{H}$	$0 < t < +\infty$	$H_0 t \approx \frac{1}{2-\beta} e^{(2-\beta)\tau}.$
BKL T	$\frac{dt}{dT} = l^3$	$-\infty < T < 0$	$H_0 l_0^3 T \approx -\frac{1}{1+\beta} e^{-(1+\beta)\tau}$
Conformal η	$\frac{dt}{d\eta} = l$	$0 < \eta < +\infty$	$H_0 l_0 \eta \approx \frac{1}{1-\beta} e^{(1-\beta)\tau}$
G_2-adapted ξ	$\frac{dt}{d\xi} = l e^{-2\beta^+}$	$0 < \xi < +\infty$	$\xi \approx \eta \quad \text{if } \frac{2}{3} < \gamma < \frac{4}{3}.$

Table C.1 Time variables for Bianchi VII$_0$ universes. Dimensionless time τ assumes all real values, $-\infty < \tau < +\infty$, and $\tau \to +\infty$ in the late time asymptotic regime. The constant β is given by $\beta = \frac{1}{2}(4 - 3\gamma)$, where γ is the equation of state parameter.

equations (3.34) and (1.3) and the fact that $\beta^+ \to 0$ as $\tau \to +\infty$ if $\frac{2}{3} < \gamma < \frac{4}{3}$ (see (C.10)). The ξ-time was apparently first used for Bianchi cosmologies by Collins & Hawking (1973a, see page 328), and subsequently by Siklos (1980). We use the name "G_2-adapted time" because the G_3 isometry group admits a two-parameter Abelian subgroup G_2, and $l e^{-2\beta^+}$ is the length scale in the preferred direction orthogonal to the orbits of the G_2.

Collins and Hawking 1973a analyzed the asymptotic evolution of Bianchi VII$_0$ universes assuming that the matter in the model consisted of two non-interacting components, dust ($\gamma = 1$) and radiation ($\gamma = \frac{4}{3}$). The dust dominates at late times. The tilt was not assumed to be zero. They proved that if the initial state is close enough to the Einstein-de Sitter model, then the Bianchi VII$_0$ model isotropizes in the sense that $\sigma/H \to 0$ and $\beta^\pm \to \beta^\pm_\infty$ as $t \to +\infty$, where $\beta^\pm_\infty$ are constants (see their Theorem 3). Prior to proving this result they heuristically determine the asymptotic form of $\beta^\pm$, finding that

$$\beta^- \approx e^{-\alpha}(A\cos 2\xi + B\sin 2\xi), \quad \beta^+ \approx Ce^{-\alpha},$$

(see page 332). To facilitate comparison we note that $e^\alpha = l$, $H = \dot\alpha$, and in their line-element in Section IV, $\lambda = -2\beta^+$, $\lambda = 2\sqrt{3}\beta^-$, $\psi = b_1 = b_2 = 0$. On using Table C.1 and noting that $l = l_0 e^\tau$, we see that their result agrees with our result (C.10), when $\gamma = 1$ i.e. $\beta = \frac{1}{2}$. It is of interest that if the field equations are linearized about the Einstein de-Sitter model they do not give the correct asymptotic form for β^+. This fact can be seen most clearly from the subsequent paper Collins and Hawking (1973b) in which they give the asymptotic form of the solution of the linear field equations for Bianchi VII$_0$ universes with dust. In terms of the conformal time η (see Table C.1),

$$\beta_{11} = -2\beta^+ \approx \frac{A}{\eta^3}, \quad \beta_{22} - \beta_{33} = 2\sqrt{3}\beta^- \approx \frac{1}{\eta^2}(B\cos 2\eta + C\sin 2\eta)$$

(see pages 316-7).

Doroshkevich et al 1973 analyzed the non-tilted Bianchi VII$_0$ universes using the line-element (C.3) and the BKL time T. For dust ($\gamma = 1$), they give the asymptotic form

$$\beta^- \approx C_1 T^{-\frac{2}{3}} \sin\left(C_2 T^{-\frac{1}{3}} + \psi_0\right), \quad l^2 e^{-2\beta^+} \approx C_3 T^{-\frac{4}{3}},$$

(see Appendix I, equation (I.20)). To facilitate comparison we note that their metric variables γ, μ and $(\lambda_1\lambda_2)^{\frac{1}{2}}$ are related to ours by $\gamma = l^6$, $\mu = 4\sqrt{3}\beta^-$ and $(\lambda_1\lambda_2)^{\frac{1}{2}} = l^2 e^{-2\beta^+}$. On using Table C.1 we see that their result for β^- agrees with our result (C.10) when $\gamma = 1$ (i.e. $\beta = \frac{1}{2}$). Their result for β^+ simply confirms that $e^{\beta^+} \to$ constant as $t \to +\infty$ but does not specify how fast.

For radiation ($\gamma = \frac{4}{3}$) Doroshkevich et al give the following asymptotic forms

$$\beta^- \approx C_1 T\theta^{-\frac{1}{2}}\sin[\psi(T)], \quad l^2 e^{-2\beta^+} \approx C_3 \frac{\theta(T)}{T^2}, \tag{C.13}$$

with

$$\frac{d\psi}{dT} = C_2 \frac{\theta(T)}{T^2}, \quad \theta(T) \approx \frac{1}{ln(\frac{C_4}{T})},$$

where C_1, C_2, C_3 and C_4 are constants (see Appendix I, equations (I.14 - I.15); we have relabelled their constants and have truncated their asymptotic expansions). In order to compare with our results we note that

$$\frac{d\psi}{d\tau} = \frac{1}{l^3 H}\frac{d\psi}{dT} \approx \frac{1}{l_0^3 H_0}e^{-\tau}\frac{d\psi}{dT},$$

as follows from Table C.1 and equation (C.10). In addition Table C.1 gives $T \sim e^{-\tau}$, suppressing constants, and hence $\theta \sim \tau^{-1}$ as $\tau \to +\infty$. It now follows that (C.13) is in agreement with our result (C.11).

Finally, Barrow and Sonoda 1986, using the equations of Lukash 1975, investigated the asymptotic behaviour of non-tilted Bianchi VII$_0$ universes with perfect fluid and γ-law equation of state with $1 < \gamma < \frac{4}{3}$. They used ξ-time, up to a factor of 2. By solving the linearized

field equations they obtained

$$\beta^- \sim \xi^{-\frac{1}{1-\beta}}(A\cos2\xi + B\sin2\xi),$$

as $\xi \to +\infty$, after compensating for the factor of 2 (see their equation (4.144)). This form agrees with our result (C.10). In order to facillitate comparison, we note that their μ is related to β^- by $\mu = 2\sqrt{3}\beta^-$, and mention that they redefine μ in their equation (4.138). Their parameter α is related to our β by $\alpha = 2\beta/(1-\beta)$. Barrow and Sonoda argue that their linearization procedure is consistent only if $\gamma \le \frac{5}{4}$, suggesting that a bifurcation occurs at $\gamma = \frac{5}{4}$. In contrast, we find that a bifurcation occurs at $\gamma = \frac{4}{3}$, and that the asymptotic form (C.10) is valid for γ between $\frac{2}{3}$ and $\frac{4}{3}$.

Acknowledgements

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